Sequences and Series

4

LEARNING OBJECTIVES

In this chapter, you learn how to use GC to

- Find the nth term of a Sequence.
- Find the Sum of a Sequence.
- Find the Sum to Infinity of a GP.
- Solve Quadratic Equation under EQUA mode.
- Evaluate Σ.
- Find the Root(s) of an Equation under GRAPH mode.
- Determine the Behaviour of a Sequence.

Example 4.1

A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero. The first three terms of the geometric series are equal to the first, sixth and tenth terms respectively of the arithmetic series.

(i) Show that $5r^2 - 9r + 4 = 0$.

(ii) Deduce that the geometric series is convergent and find, in terms of a, the sum to infinity.

(iii) The sum of the first n terms of the geometric series is denoted by S. Given that $a > 0$, find the least value of n for which S exceeds 99% of the sum to infinity.

SOLUTION

(i) $r=\dfrac{a+(6-1)d}{a}=\dfrac{a+(10-1)d}{a+(6-1)d}$

$$r=1+5\left(\frac{d}{a}\right)=\frac{1+9\left(\frac{d}{a}\right)}{1+5\left(\frac{d}{a}\right)}$$

If x, y and z are in G.P., $\dfrac{y}{x}=\dfrac{z}{y}=r$

$$\therefore\frac{d}{a}=\frac{r-1}{5}\Rightarrow r=\frac{1+9\left(\frac{r-1}{5}\right)}{1+5\left(\frac{r-1}{5}\right)}\Rightarrow r^2=1+9\left(\frac{r-1}{5}\right)\Rightarrow 5r^2=5+9r-9$$

$\Rightarrow 5r^2-9r+4=0$ (shown)

(ii) $5r^2-9r+4=0\ \Rightarrow (r-1)(5r-4)=0\Rightarrow r=1$ or $r=\dfrac{4}{5}$

Since $d\neq 0,$ the three terms of GP are not the same, thus $r\neq 1$.

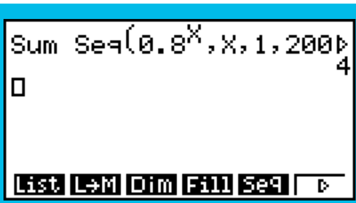
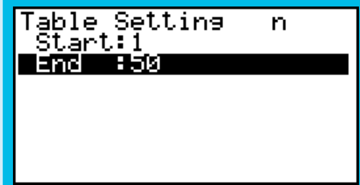
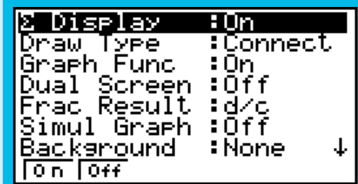
Hence, $r=\dfrac{4}{5}\Rightarrow$ Since $|r|<1,$ the geometric series is convergent.

And the sum to infinity $=\dfrac{a}{1-\frac{4}{5}}=5a.$

(iii)
$$S_n>0.99S_\infty$$
$$\frac{a(1-0.8^n)}{1-0.8}>0.99\frac{a}{1-0.8}$$
$$1-0.8^n>0.99$$
$$0.8^n<0.01$$
$$n\lg 0.8<\lg 0.01$$
$$n>\frac{\lg 0.01}{\lg 0.8}$$
$$n>20.6$$

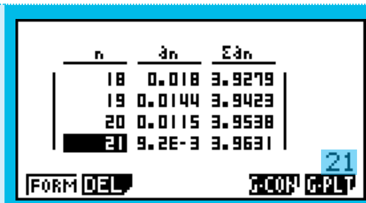
Hence, the least value of n is 21.

(You may use GC to verify your answer.)

Keystrokes	Screen	Explanation
MENU 1		Select **RUN.MAT** mode.
OPTN F1		Select **LIST**.
F6 F6 F1		Select **SUM**.
F6 F5		Select **Seq**.
0 • 8 ∧ X,θ,T ▶ , X,θ,T , 1 , 2 0 0 , 1 tan EXE		Enter the parameters in the following format: ***Seq*** *(Expression, Variable, Start, End, Increment)*
		Answer: "4" which is S_∞ where $a = 0.8$ and $r = 0.8$. (Here, we assume a takes a positive value.)
MENU 8		Select **RECUR** mode.
F3 F1		Select **TYPE**, F1: $a_n = An + B$.
0 • 8 ∧ X,θ,T EXE		Enter the expression 0.8^n.
F5		Select **SET**.
1 EXE 5 0 EXE EXE		Enter "1" for Start, "50" for End. Increase the End value if found necessary later.
SHIFT MENU		Enter **SET UP**.
F1 EXE		Set "On" for $\sum$Display.

F6

▼ …



Select **TABL.**

Press ▼ until $\sum a_n$ exceeds $0.99S_\infty$, i.e. $0.99 \times 4 = 3.96$.

You may also consider 0.8^n falling below 0.01.

Both approaches arrive at the same answer: the least value of n is 21. ☑

Example 4.2

(i) Patrick saves \$20 on 1 January 2008. On the first day of each subsequent month he saves \$4 more than in the previous month, so that he saves \$24 on 1 February 2008, \$28 on 1 March 2009, and so on. On what date will he first have saved over \$5000 in total?

(ii) Kenny puts \$20 on 1 January 2008 into a bank account which pays compound interest at a rate of 3% per month on the last day of each month. He puts a further \$20 into the account on the first day of each subsequent month.

(a) How much compound interest has his original \$20 earned at the end of 3 years?

(b) How much in total, correct to the nearest dollar, is in the account at the end of 3 years?

(c) After how many complete months will the total in the account first exceed \$5000?

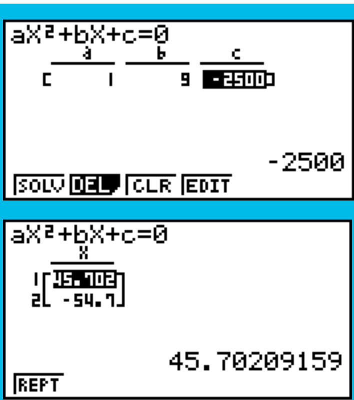
Solution

(i) $T_1 = 20,\ d = 4$

$$\frac{n}{2}\left[2(20)+4(n-1)\right] > 5000 \Rightarrow n^2 + 9n - 2500 > 0 \Rightarrow n < -54.7 \text{ or } n > 45.7$$

∴ Hence, Patrick will first have saved over $5000 in total on 1 October 2011.

(You may use GC to find the roots.)



MENU X,θ,T	Select **EQUA** mode.
F2	Select **F2:Polynomial.**
F1	Select **Degree 2.**
1 EXE 9 EXE (−) 2 5 0 0 EXE	Enter the values of a, b and c.
F1	Select **SOLV.**
	Answers: 45.7 or −54.7

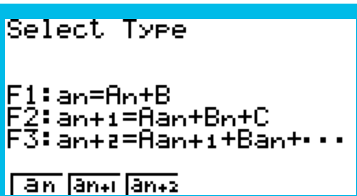
(ii) (a) The required compound interest $= 20(1.03)^{36} - 20 = \$38.0$ (3 s.f.)

(b) The required amount $= 20(1.03) + 20(1.03)^2 + \ldots + 20(1.03)^{36}$

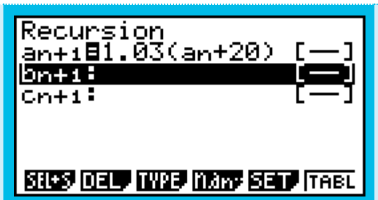
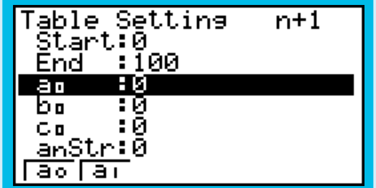
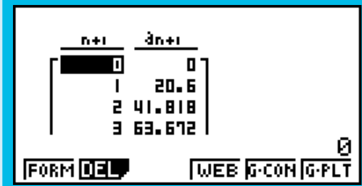
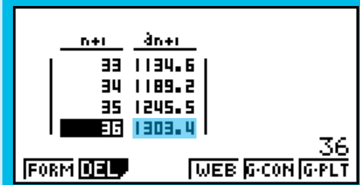
$$= \frac{20(1.03)\left[(1.03)^{36} - 1\right]}{1.03 - 1}$$

$= \$1303$

(You may use GC to verify your answer.)



MENU 8	Select **RECUR** mode.
F3	Select **TYPE.**
F2	Select F2: $a_{n+1} = Aa_n + Bn + C$.

1 · 0 3 cos F2 + 2 0 tan EXE		Enter the recurrence formula. Press F2 for a_n.
F5 ▼ 1 0 0 EXE		Select **SET.** Enter the start and end value of n as well as the value of a_n.
EXIT F6		Select **TABL.**
▼ ...		Keep pressing ▼ until the value on the left column reaches 36. Record the corresponding value on the right column: 1303.4. ☑

(c) $20(1.03)+20(1.03)^2+...+20(1.03)^n > 5000$

$$\frac{20(1.03)(1.03^n-1)}{1.03-1} > 5000$$

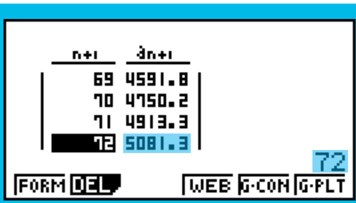
$$1.03^n > 8.282$$

$$n > \frac{\lg 8.282}{\lg 1.03}$$

$$n > 71.5$$

The total in the account will first exceed \$5000 after 72 months.

(You may use GC to verify your answer.)

▼ …		Keep pressing ▼ until the value on the right column first reaches 5000 or above. Record the corresponding value on the right column: 72. ☑

Example 4.3

A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and

$$u_{n+1} = u_n - \frac{3n^2 + 3n + 1}{n^3(n+1)^3}, \text{ for all } n \geq 1.$$

(i) Use the method of mathematical induction to prove that $u_n = \dfrac{1}{n^3}$.

(ii) Hence find $\displaystyle\sum_{n=1}^{N} \frac{3n^2 + 3n + 1}{n^3(n+1)^3}$.

(iii) Give a reason why the series in part (ii) is convergent and state the sum to infinity.

(iv) Use your answer to part (ii) to find $\displaystyle\sum_{n=2}^{N} \frac{3n^2 - 3n + 1}{n^3(n-1)^3}$.

Solution

(i) Let P_n be the statement $u_n = \frac{1}{n^3}$ where $u_{n+1} = u_n - \frac{3n^2+3n+1}{n^3(n+1)^3}$ and $u_1 = 1$ for $n \in \mathbb{Z}^+$.

When $n = 1$, L.H.S. $= u_1 = 1$; R.H.S. $= \frac{1}{1^2} = 1$.

$\because$ L.H.S. = R.H.S.

$\therefore$ P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \frac{1}{k^3}$.

To prove P_{k+1} is true, i.e. $u_{k+1} = \frac{1}{(k+1)^3}$

$$\begin{aligned}
\text{L.H.S.} = u_{k+1} &= u_k - \frac{3k^2+3k+1}{k^3(k+1)^3} \\
&= \frac{1}{k^3} - \frac{3k^2+3k+1}{k^3(k+1)^3} \\
&= \frac{(k+1)^3 - 3k^2 - 3k - 1}{k^3(k+1)^3} \\
&= \frac{k^3+3k^2+3k+1-3k^2-3k-1}{k^3(k+1)^3} \\
&= \frac{1}{(k+1)^3} \\
&= \text{R.H.S.}
\end{aligned}$$

$\therefore$ P_{k+1} is true whenever P_k is true.

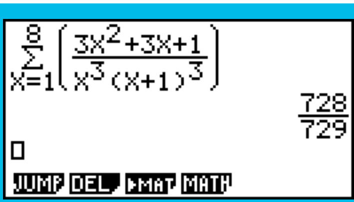
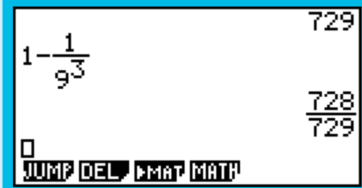
Since P_1 is true and $P_k \Rightarrow P_{k+1}$ is true by Mathematical Induction,

P_n is true for all $n \in \mathbb{Z}^+$.

(ii)
$$\begin{aligned}
\sum_{n=1}^{N} \frac{3n^2+3n+1}{n^3(n+1)^3} &= \sum_{n=1}^{N} (u_n - u_{n+1}) \\
&= u_1 - u_2 \\
&\quad + u_2 - u_3 \\
&\quad \dots \\
&\quad + u_{N-1} - u_N \\
&\quad + u_N - u_{N+1} \\
&= u_1 - u_{N+1} \\
&= 1 - \frac{1}{(N+1)^3}
\end{aligned}$$

UNIT 4 SEQUENCES AND SERIES

(You may use GC to verify your answer.)

[MENU] [1] [F4] [F6] [F2] [▼] [◀] [X,θ,T] [▶] [1] [▲] [8] [▼] [◀] [◀] [◀] [◀] [$a\frac{b}{c}$] [3] [X,θ,T] [x^2] [+] [3] [X,θ,T] [+] [1] [▼] [X,θ,T] [^] [3] [▶] [cos] [X,θ,T] [+] [1] [tan] [^] [3] [EXE]		Select **RUN.MAT** mode. Select **MATH**, $\sum$(. Enter the initial value "1" and the end value "8" (randomly chosen). Enter the expression. **Answer:** 728/729. If the answer is not in fraction form, press [F↔D].
[1] [−] [$a\frac{b}{c}$] [1] [▼] [9] [^] [3] [EXE]		Evaluate the answer in part (ii) when $N = 8$. **Answer:** 728/729 ☑

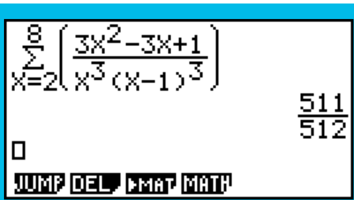
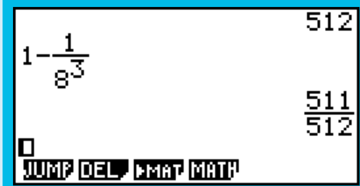
(iii) When $N \to \infty, \frac{1}{(N+1)^3} \to 0 \Rightarrow 1 - \frac{1}{(N+1)^3} \to 1 \Rightarrow \sum_{n=1}^{N} \frac{3n^2+3n+1}{n^3(n+1)^3} \to 1$

Hence, the series in part (ii) is convergent and the sum to infinity is 1.

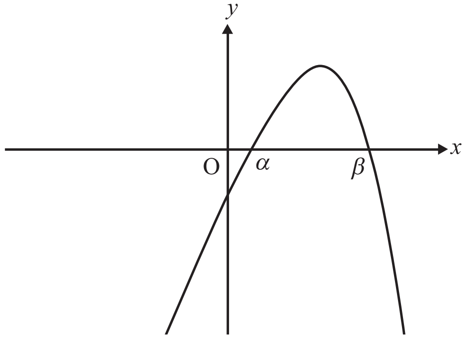
(iv)

$$\begin{aligned}\sum_{n=2}^{N}\frac{3n^2-3n+1}{n^3(n-1)^3} &= \sum_{r=1}^{N-1}\frac{3(r+1)^2-3(r+1)+1}{(r+1)^3r^3} \quad \text{when } n=r+1\\ &= \sum_{r=1}^{N-1}\frac{3(r+1)^2-3(r+1)+1}{r^3(r+1)^3}\\ &= \sum_{r=1}^{N-1}\frac{3r^2+6r+3-3r-3+1}{r^3(r+1)^3}\\ &= \sum_{r=1}^{N-1}\frac{3r^2+3r+1}{r^3(r+1)^3}\\ &= 1-\frac{1}{(N-1+1)^3}\\ &= 1-\frac{1}{N^3}\end{aligned}$$

(You may use GC again to verify your answer.)

[Apply similar keystrokes found in part (ii).]		Answer: 511/512
		Answer: 511/512 ☑

Example 4.4



The diagram shows the graph of $y = 2x - e^{\frac{x}{2}}$. The two roots of the equation are denoted by α and β. where $\alpha < \beta$.

(i) Find the values of α and β, each correct to 3 decimal places.

A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \frac{1}{2}e^{\frac{x_n}{2}}$$

for $n \geq 1$.

(ii) Prove algebraically that, if the sequence converges, then it converges to either α or β.

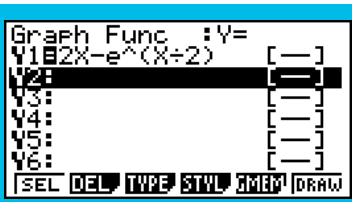
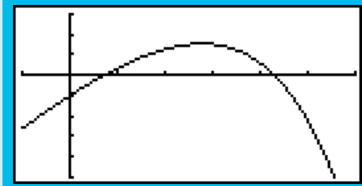
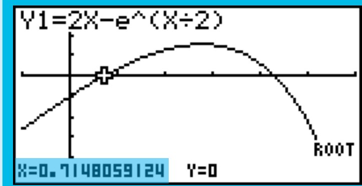
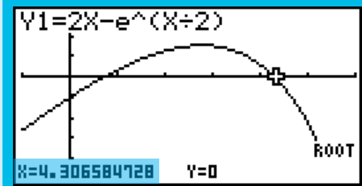
(iii) Use a calculator to determine the behaviours of the sequence for each of the cases $x_1 = 0$, $x_1 = 3$, $x_1 = 6$.

(iv) By considering $x_{n+1} - x_n$, prove that

$$x_{n+1} < x_n \text{ if } \alpha < x_n < \beta,$$
$$x_{n+1} > x_n \text{ if } x_n < \alpha \text{ or } x_n > \beta,$$

(v) State briefly how the results in part (iv) relate to the behaviours determined in part (iii).

Solution

[MENU] [5] [2] [X,θ,T] [−] [SHIFT] [ln] [cos] [X,θ,T] [÷] [2] [tan] [EXE]		Enter **GRAPH** mode. Enter the equation of the graph as Y1.
[F6]		Select **DRAW**.
[F5] [F1]		Find the first root α. Answer: 0.715
▶		Find the second root β. Answer: 4.307

(i) From GC, $\alpha = 0.715$ and $\beta = 4.307$.

(ii) $x_{n+1} = \frac{1}{2} e^{\frac{x_n}{2}}$

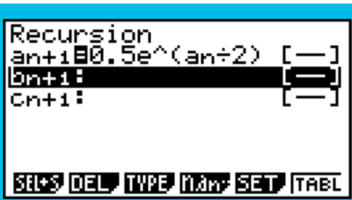
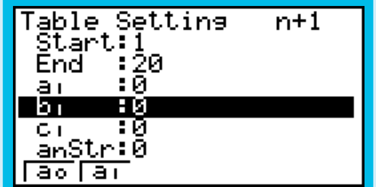
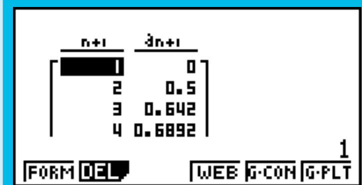
$\Rightarrow 2x_{n+1} = e^{\frac{x_n}{2}}$

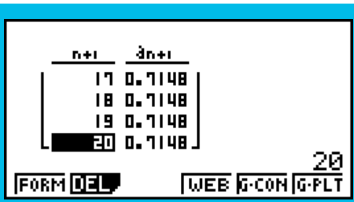
$\Rightarrow 2x_{n+1} - e^{\frac{x_n}{2}} = 0$

$\Rightarrow 2L - e^{\frac{L}{2}} = 0$ given that $x_n \to L$ and $x_{n+1} \to L$ when $n \to \infty$.

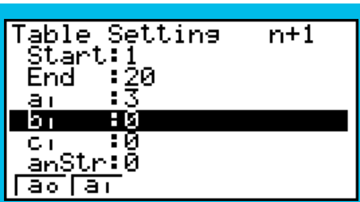
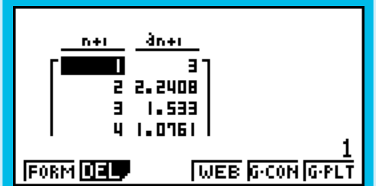
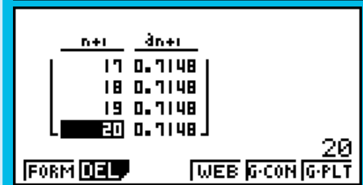
Since α and β are the roots of $2x - e^{\frac{x}{2}} = 0$, hence x_n converges to α or β if the sequence converges.

(iii)

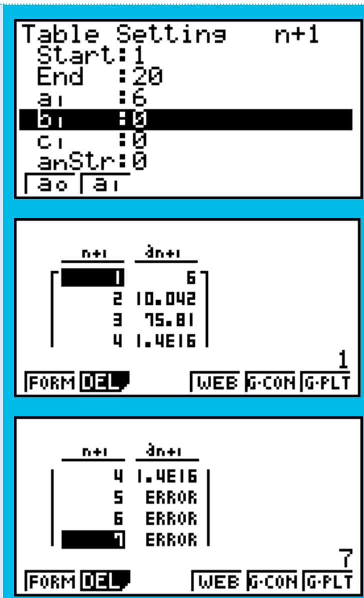
[MENU] [8] [F3] [F2] [0] [•] [5] [SHIFT] [ln] [cos] [F2] [÷] [2] [tan] [EXE]		Enter **RECUR** mode. Select **TYPE**. Select F2: $a_{n+1} = Aa_n + B_n + C$. Enter the recurrence equation. [F2] corresponds to a_n.
[F5] [F2] [1] [EXE] [2] [0] [EXE] [0] [EXE]		Enter Table Setting. Change a_0 to a_1. Apply the settings as shown. Use a larger value for "End" if later found insufficient. Use $a_1 = 0$ for the first case.
[EXIT] [F6]		Select **TABL**.

⊙…		While pressing ⊙, observe the change in a_{n+1}. It is noted that the sequence converges to 0.7148 which is α when $x_1 = 0$.

Repeat the same process for the cases $x_1 = 3$ and $x_1 = 6$.

[Set $a_1 = 3$ and apply similar key strokes.]	  	It is noted that the sequence converges to 0.7148 also which is α when $x_1 = 3$.

[Set $a_1 = 6$ and apply similar key strokes.]



It is noted that the sequence diverges when $x_1 = 6$ as shown by the 'ERROR' message.

(iv) $x_{n+1} - x_n = \frac{1}{2}e^{\frac{x_n}{2}} - x_n = \frac{1}{2}e^{\frac{x_n}{2}} - x_n = -\frac{1}{2}(2x_n - e^{\frac{x_n}{2}})$

If $\alpha < x_n < \beta$, $2x_n - e^{\frac{x_n}{2}} > 0 \Rightarrow -\frac{1}{2}(2x_n - e^{\frac{x_n}{2}}) < 0 \Rightarrow x_{n+1} - x_n < 0 \Rightarrow x_{n+1} < x_n$.

If $x_n < \alpha$ or $x_n > \beta$, $2x_n - e^{\frac{x_n}{2}} < 0 \Rightarrow -\frac{1}{2}(2x_n - e^{\frac{x_n}{2}}) > 0 \Rightarrow x_{n+1} - x_n > 0 \Rightarrow x_{n+1} > x_n$.

(v) For $x_1 = 0$ where $x_1 < \alpha \Rightarrow x_{n+1} = \frac{1}{2}e^{\frac{x_n}{2}} - x_n < \frac{1}{2}e^{\frac{\alpha}{2}} = \alpha$, hence $x_1 < x_2 < x_3 < ... < \alpha$.

For $x_1 = 3$ where $\alpha < x_1 < \beta \Rightarrow x_n > x_{n+1} = \frac{1}{2}e^{\frac{x_n}{2}} > \frac{1}{2}e^{\frac{\alpha}{2}} = \alpha$, hence $x_1 > x_2 > x_3 > ... > \alpha$.

If $x_1 = 6$ where $x_1 > \beta \Rightarrow x_n < x_{n+1}$, hence $\beta < x_1 < x_2 < x_3 ...$.

Checklist: GC Techniques covered in this chapter

TECHNIQUES	4.1	4.2	4.3	4.4
Find the **nth term of a Sequence**.	☑	☑		
Find the **Sum of a Sequence**.	☑			
Find the **Sum to Infinity of a GP**.	☑			
Solve **Quadratic Equation** under **EQUA** mode.		☑		
Evaluate Σ.			☑	
Find the **Root(s) of an Equation** under **GRAPH** mode.				☑
Determine the **Behaviour of a Sequence**.				☑